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Analytical correlation of indentation experiments

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ABSTRACT

We improve the analysis of indentation by the model of the expansion of a spherical cavity by considering a generalisation of the velocity field proposed by Avitzur for metal extrusion: its components vary with the distance to the indenter apex r as $1/r^s$ with s a free parameter. By neglecting the material displacement we apply this model to conical indentation (apical angle 2θ) of an elastic solid (Young's modulus E), a rigid perfectly plastic (RPP) solid (yield stress σ_0) and elastic perfectly plastic (EPP) solids with indentation index $X = (E^*/\sigma_0)\cot\theta$ and estimate the hardness H and the shape ratio $c = h_c/h$ where h and h_c are the penetration depth and the contact depth. s is determined by minimising the total work (elastic case) and the plastic power (RPP and EPP cases). The results of the model are in agreement with the available exact results (elastic case) and numerical ones (RPP and EPP cases).

§ 1. INTRODUCTION

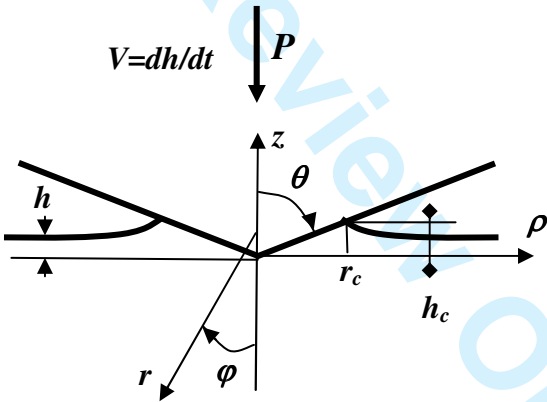


Figure 1. Axisymmetric indenter profile (here the cone $p = 1$).

We consider the indentation of an homogeneous solid by an axisymmetric indenter with a power law profile (figure 1):

$$h(z) = \frac{\rho^p}{pR^{p-1}} \rightarrow h_c = \frac{r_c^p}{pR^{p-1}} \tag{1}$$

r_c is the contact radius and h_c the contact depth. The sphere (radius R) corresponds to $p = 2$; the flat punch corresponds to $p \infty$ and a constant contact radius r_c . The cone with the apical

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$$\text{the shape ratio } c^p = \frac{h_c}{h} \quad \text{the reduced hardness } H^* = \frac{H}{\sigma_0} \quad (2)$$
$$X = \frac{E^*}{\sigma_0} \tan \beta \quad (3)$$
$$\mathbf{u} \left\{ \begin{array}{l} u_r = V \left(\frac{r_c}{r} \right)^s \cos \varphi \\ u_\varphi = V \left(\frac{s-2}{2} \right) \left(\frac{r_c}{r} \right)^s \sin \varphi \\ u_\psi = 0 \end{array} \right. \quad \text{with } V = \frac{dh}{dt} \quad (r > r_c) \quad (4)$$

For $s = 1$, it is the velocity field induced in an elastic half space by a concentrated normal force at $r = 0$ [2]. For $s = 2$, it is the radial field proposed by Avitzur for describing axisymmetric metal extrusion or drawing [7]. It is incompressible and induces a velocity discontinuity at $r = r_c$ with a dead zone below the indenter (where we assume $\mathbf{u} = \mathbf{V}$); we do not take into account this discontinuity further. The decrease rate of the matter velocity with the distance to the indenter depends on the value of the exponent s . So we can obtain the various possible cases:

- sink-in for $s < 2$ ($u_\varphi(r, \pi/2) < 0$)
- pile-up for $s > 2$ ($u_\varphi(r, \pi/2) > 0$).

By neglecting the material displacement (asymptotic model valid if $\tan\beta \ll 1$ and/or high friction [8]), it is possible to derive the value of the shape ratio c^p which takes the following form for the present velocity field:

$$c^p = 1 + p \int_0^{r_c} \left(\frac{s-2}{2} \right) \left(\frac{r_c(t)}{r_c} \right)^{s+p-1} \frac{dr_c(t)}{r_c} \quad (5)$$

$r_c(t)$ is the current value of the contact radius and r_c its final value. For the limiting cases (elastic and RPP solids), s does not depend on $r_c(t)$ and the shape ratio is related simply to the values of s and p :

$$c^p = \frac{s}{2} \left(\frac{p+2}{p+s} \right) \quad (6)$$

For conical indenter, geometrical similarity insures that the shape ratio c and the reduced hardness H^* are in all cases constant. In the EPP case the ratio $B = b/r_c$ of the elastic-plastic boundary radius b to the contact radius r_c is constant and in each zone we describe the velocity field by a constant value of s . So for conical indentation of EPP solids, the calculation of (5) must be performed with a value $s_e = 1$ for $r_c(t) < r_c/B$ (elastic zone, see § 3.1) and another value of s for $r_c/B < r_c(t) < r_c$ (elastoplastic zone):

$$c = \frac{3}{2(s+1)} \left(s + \frac{1-s}{2B^{s+1}} \right) \quad (7)$$

2.2. The strain rate field and the strain level

Application of the standard formula [9] provides easily the non zero strain rates:

$$\begin{cases} \dot{\epsilon}_r = -s \left(\frac{V}{r_c} \right) \left(\frac{r_c}{r} \right)^{s+1} \cos \varphi \\ \dot{\epsilon}_\varphi = \dot{\epsilon}_\psi = \frac{s}{2} \left(\frac{V}{r_c} \right) \left(\frac{r_c}{r} \right)^{s+1} \cos \varphi \\ \dot{\epsilon}_{r\varphi} = \frac{s(1-s)}{4} \left(\frac{V}{r_c} \right) \left(\frac{r_c}{r} \right)^{s+1} \sin \varphi \end{cases} \quad (8)$$

So the second invariant of the strain rate field is:

$$\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij} = \frac{3}{2} \left(s \left(\frac{V}{r_c} \right) \left(\frac{r_c}{r} \right)^{s+1} \right)^2 \left(\cos^2 \varphi + \frac{(s-1)^2}{12} \sin^2 \varphi \right) \quad (8)$$

For the case where s is constant and uniform, we can estimate the generalised strain [9]:

$$\bar{\varepsilon}(r) = \int_0^t \left(\int_0^{\pi/2} \dot{\varepsilon} \sin \varphi d\varphi \right) dt = \int_0^t \left(\int_0^{\pi/2} \sqrt{\frac{2}{3}} \dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij} \sin \varphi d\varphi \right) dt = \frac{2F(s)}{p+2} \left(\frac{r_c}{r} \right)^{s+1} \tan \beta \quad (9)$$

where $F(s) = \frac{1}{2} \left(1 + \frac{S}{\sqrt{1-S}} \ln \left(\frac{1+\sqrt{1-S}}{\sqrt{S}} \right) \right)$ with $S = \frac{(s-1)^2}{12} < 1$

$F(s)$ is a slowly increasing function of s with $F(1) = 0.5$ and $F(4.46) \sim 1$; the time integration is in fact an integration versus h which is transformed in an integration versus $r_c(t)$ by using (6), (2) and (1). We observe that as r increases the strain decreases very strongly if $s \sim 4$ (RPP case, see § 3.2). The equation (9) is in good agreement with empirical estimations of the representative strain for hardness of metals measured with cones [11] and spheres [10]: the strain increases in direct relation with $\tan \beta$ and for $s \sim 4$ (plastic case) its maximal value (at $r = r_c$) is about 2.2-2.5 times higher than the representative strains: $0.2r_c/R$ or $0.3\cot \theta$.

§ 3. THE LIMITING CASES

3.1. Indentation of an elastic material

First we consider the flat punch. Because the contact radius r_c is constant, the strain field ε is given by (8) in which we change V into h . The deviatoric stress tensor is $s = 2\mu\varepsilon$ where $\mu = E/[2(1+\nu)]$ is the elastic shear modulus. Because the velocity field is incompressible, the indentation work is ($s > 1/2$):

$$W_e(s) = 2\pi\mu \int_{r_c}^{\infty} \left(\int_0^{\pi/2} \varepsilon_{ij}\varepsilon_{ij} \sin \varphi d\varphi \right) r^2 dr = \pi\mu r_c h^2 \left(\frac{s^2}{(2s-1)} \left(1 + \frac{(s-1)^2}{6} \right) \right) \quad (10)$$

Because the displacement field solution minimises the work, the best estimation of s is obtained by minimising W_e with respect to s : this gives easily $s = s_e = 1$: the strain field of the concentrated normal force is so recovered. So we can derive the force and its variation with h :

$$\begin{cases} P = \frac{\partial W_e}{\partial h} = 2\mu\pi r_c h = \delta(2E^* r_c h) \\ \frac{dP}{dh} = \delta(2E^* r_c) \end{cases} \quad \text{with} \quad \delta = \frac{\pi(1-\nu)}{2} \quad (11)$$

The difference with the exact result [2] is the factor $\delta \sim 1.09$ for $\nu = 0.3$; so the error is 9 % about. The general case (p finite) is obtained by assuming $s = 1$ and by integrating dP/dh and the strain field versus h [12], approach which is valid for zero friction and adhesion. As h is proportional to r_c^p (relations (1)-(2)), we obtain:

$$P = \int_0^h \delta 2 E^* r_c(t) dh(t) = \delta 2 E^* \int_0^{r_c} r_c(t) \frac{r_c^{p-1}(t)}{c^p R^{p-1}} dr_c(t) = \delta \left(\frac{p}{1+p} \right) 2 E^* r_c h \tag{12}$$

The von Mises yield criterion $s_{ij}s_{ij} = 4\mu^2 \varepsilon_{ij}\varepsilon_{ij}$ is maximal at $(r = r_c, \varphi = 0)$ and yield occurs as it attains $2\sigma_0^2/3$ [9] for the value of the indentation index X_0 and the mean contact pressure H_0 :

$$X_0 = \frac{E^*}{\sigma_0} \tan \beta = \frac{1}{\delta} \left(1 + \frac{p}{2} \right) \quad \text{and} \quad H_0 = \frac{P_0}{\pi r_{c0}^2} = \frac{2\sigma_0}{3} \tag{13}$$

Comparison of the results of the present model with the exact results (table 1) demonstrates that the shape ratio is overestimated and the contact pressure underestimated. Such a discrepancy is expected because the exact solution is much more complex, but it is not very important because our aim is to study the EPP indentation. Yield occurs for the same pressure than for the classical expanding cavity model [4].

Table 1. Indentation of an elastic solid: Comparison of the exact results [2] with the results of the present model ($\nu = 0.3$).

p	1 (cone ; $\tan\beta = \cot\theta$)			2 (sphere ; $\tan\beta = r_c/R$)		
Exact	c	$H/(E^*\tan\beta)$		c^2	$H/(E^*\tan\beta)$	X_0
results	$2/\pi \sim 0.63$	0.5		0.5	$(4/3\pi) \sim 0.42$	~ 2.6
Model ($s = 1$)	0.75	0.467		0.66	0.35	1.8

3.2. Indentation of a RPP solid

The plastic power [9] is (by assuming $s > 2$):

$$\dot{W}_p(s) = 2\pi \int_{r_c}^{\infty} \left(\int_0^{\pi/2} \sigma_0 \dot{\varepsilon} \sin \varphi \, d\varphi \right) r^2 dr = V \pi r_c^2 \sigma_0 \left(\frac{2sF(s)}{s-2} \right) \tag{14}$$

The velocity field solution minimises the power [9]; so the best estimation is obtained by minimising the power with respect to s ; this gives $s = s_p \sim 4.35$ and the hardness $H = \dot{W}_p/(\pi r_c^2 V) \sim 3.62\sigma_0$. These results apply to all indenter shapes. Such a value of the exponent s corresponds to a quasi-radial distribution of strain rate and strain (see equation (8)) as observed in indentation of workhardened metals [4]. We see on table 2 that the present model provides values of the shape ratio and the hardness very near the values obtained with the asymptotic approach under sticking friction.

Table 2. Indentation of a RPP solid: Comparison of the results of the asymptotic analysis [8] and the results of the present model.

p	1 (cone)	2 (sphere)	$H^*=H/\sigma_0$
	c	c^2	
Zero friction	1.263	1.44	3.05
Sticking friction	1.209	1.334	3.21
Model ($s = 4.35$)	1.219	1.37	3.62

§ 4. INDENTATION OF EPP SOLIDS

4.1. The velocity field

In the plastic zone ($r_c < r < b$), the velocity field is given by (4) with an exponent s . According to the results of §3.1 we assume that the elastic zone imposes at the elastic-plastic boundary a radial stress $\sigma_{rr}(r = b) = -2\sigma_0/3$. In the elastic zone ($r > b = Br_c$), the velocity field has an exponent $s = s_e = 1$. **Because** the normal velocity must be continuous for $r = b$, the velocity field is:

$$\begin{cases} u_r = V \left(\frac{r_c}{b} \right)^s \left(\frac{b}{r} \right) \cos \varphi \\ u_\varphi = -\frac{V}{2} \left(\frac{r_c}{b} \right)^s \left(\frac{b}{r} \right) \sin \varphi \\ u_\psi = 0 \end{cases} \quad r > b \quad (15)$$

The value of b is obtained by writing that the yield criterion attains its critical value at $r = b$ (and $\varphi = 0$) as the contact radius $r_c(t)$ is equal to the present value r_c . So we obtain for all indenter shapes, with $B(t)$ the current value of B **and t_0 the time for which yield occurs**:

$$\frac{1}{b^2} \int_{t_0}^t B(t)^{1-s} r_c(t) dh(t) = \frac{1}{\delta} \frac{\sigma_0}{E^*} \rightarrow \int_{t_0}^t B(t)^{1-s} r_c(t) dh(t) = \frac{1}{\delta} \frac{\sigma_0}{E^*} B^2 r_c^2 \quad (16)$$

Resolution is easy only for the flat punch (r_c constant) or the cone. For the cone, c , s and B are constant **and $t_0 = 0$** ; so we obtain easily the value of B versus the indentation index:

$$X = \frac{E^*}{\sigma_0} \cot \theta = \frac{2c}{\delta} B^{1+s} \quad \text{or} \quad B = \left(\frac{\delta X}{2c} \right)^{\frac{1}{1+s}} \quad (17)$$

The corresponding shape ratio c is given by (7).

4.2. Evolution of the velocity field and the hardness with the plastic zone radius

The exponent s in the plastic zone is obtained by minimising the effective power [9]:

$$\dot{W}_p(s, B) = 2\pi \int_{r_c}^b \left(\int_0^{\pi/2} \sigma_0 \dot{\epsilon} \sin \varphi d\varphi \right) r^2 dr - 2\pi b^2 \int_0^{\pi/2} \sigma_{rr} u_r(b, \varphi) \sin \varphi d\varphi \quad (18)$$

So we obtain:

$$\dot{W}_p(s, B) = \begin{cases} V\pi r_c^2 \sigma_0 \left(\left(\frac{2s}{s-2} \right) (1 - B^{2-s}) F(s) + \frac{2}{3} B^{2-s} \right) & \text{if } s \neq 2 \\ V\pi r_c^2 \sigma_0 \left(4F(2) \ln B + \frac{2}{3} \right) & \text{if } s = 2 \end{cases} \quad (19)$$

The hardness can be estimated as: $H = \dot{W}_p / (\pi r_c^2 V)$ by using the calculated value of s . Such results apply to all indenter shapes. Numerical calculations reveal three domains of variation of s and H with B (figure 2):

- For $1 < B \leq 1.5$, the power increases steadily with s for $s \geq 1$: so we assume $s = 1$. This domain corresponds to a progressive extension of the plastic zone starting from the initial yielding at a point. Integration of (16) is possible for all indenter shapes. X increases by a factor 2.25 and the indentation hardness increases almost linearly with X as in the elastic regime according to the equations:

$$\begin{aligned} 1 \leq B \leq 1.5 \rightarrow H &= \sigma_0 \left(\frac{5B}{3} - 1 \right) \leq 1.5 \sigma_0 \\ \begin{cases} X = \frac{1}{\delta} \left(1 + \frac{p}{2} \right) B^2 \leq \frac{2.25}{\delta} \left(1 + \frac{p}{2} \right) & p \text{ finite} \\ X = \frac{B^2}{\delta} \leq \frac{2.25}{\delta} & \text{flat punch} \end{cases} \end{aligned} \quad (20)$$

- For $1.5 \leq B \leq 3.5$, the power has a minimum for a value of s which increases almost linearly with B from 1 to 3.11. So H increases with B in a parabolic manner from 1.5 to $3.32 \sigma_0$, a value 10 % lower than the value of the RPP case (§ 3.2).
- For $B > 3.5$, a minimum exists. H increases slower and tends toward the RPP value whereas the increase in s continues before becoming very slow: values very near the limiting value 4.35 of the RPP solids are attained for values of the indentation index very much higher than the values related to real materials; so this range is not included on figure 2.

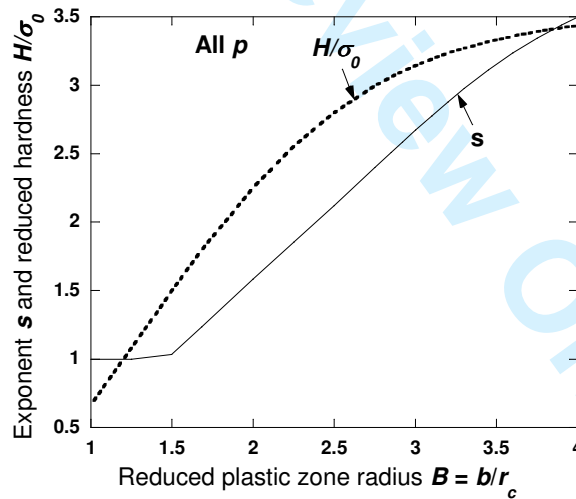


Figure 2. Evolution of the reduced hardness H/σ_0 and the plastic velocity field exponent s versus the reduced radius of the plastic zone $B = b/r_c$ (all indenter shapes).

The expanding phase of the plastic zone has no mathematical limit; so we consider only the practical conditions which correspond to $B < 4$. The evolution of s with B can be described by a second order polynomial regression:

$$1.5 \leq B \leq 4 \quad s \approx -1.073 + 1.5235B - 0.09262B^2 \quad (21)$$

4.3. Application to conical indentation

In conclusion, the analysis is now complete: for a given value of B , equations (20) or (21) provide s ; then (7) gives c ; (17) furnishes X and finally (19) gives H^* . We see on figure 3 that in first approximation the reduced plastic zone radius increases linearly with the logarithm of the indentation index, so at a rate which is significantly lower than the rate predicted by the classical cavity expansion model $B \sim (X/3)^{1/3}$ [4]:

$$B \approx 0.92817 + 0.99329 \log(X) \quad 1.5 \leq X \leq 1000 \quad (22)$$

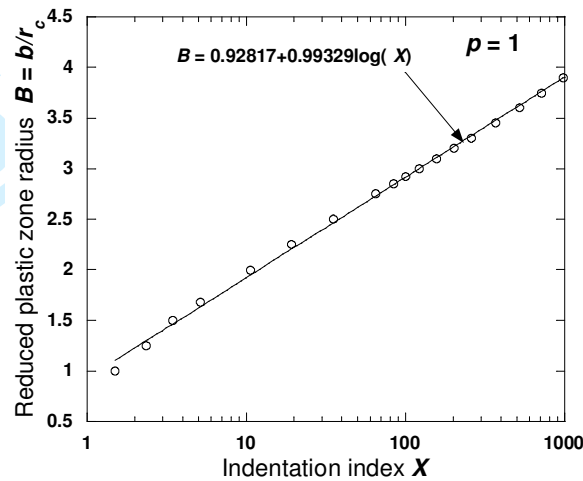


Figure 3. Evolution of the reduced radius of the plastic zone $B = b/r_c$ versus the indentation index X for conical indentation ($p = 1$).

On the figure 4 are reported the evolutions of c and H^* versus X according to the present model and the finite element calculations of Ramond-Ang         [5] related to the cone with $\theta = 70.3$ deg, equivalent to the Vickers and Berkovich pyramids. These calculations have been performed for zero friction or the maximal friction shear stress: these conditions correspond respectively to the values of Tresca's friction coefficient $\bar{m} = 0$ and 1; according to von Mises yield criterion the friction shear stress τ is given by the relation [9]:

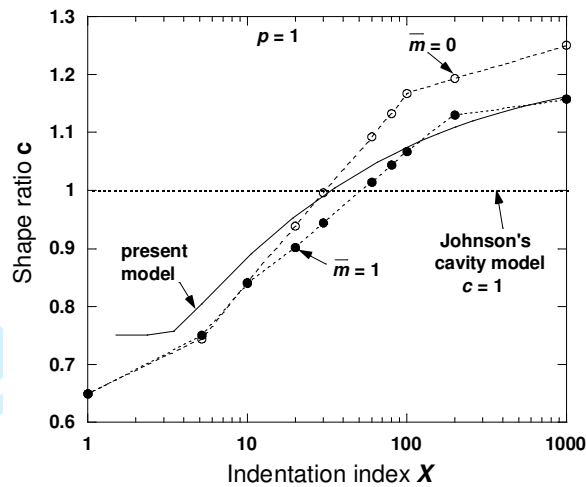
$$\tau = \bar{m} \frac{\sigma_0}{\sqrt{3}} \quad 0 \leq \bar{m} \leq 1 \quad (23)$$

From the figure 4 we can draw the following conclusions:

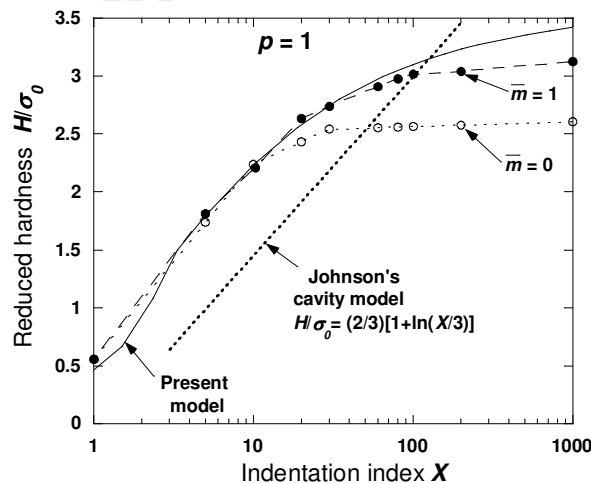
- The shape factor is first overestimated for low values of X , because the elastic contact geometry is not very well described (see table 1); then it tends for high values of X as expected toward the values related to the indentation performed under maximal friction.
- For $X < 100$ the hardness is very near the value related to maximal friction, and this corresponds to a significant improvement with respect to the Johnson's cavity model [4]. But after it becomes too high because in the RPP limit the hardness is overestimated with a difference of 12.7 % (see table 2).

We must notice that this comparison is not very easy because we do not know the values of c for $1 < X < 5$. In addition it would be interesting to know the results related to higher values of θ for which the validity of the assumption of negligible displacements is better. Nevertheless the model is able to describe continuously the transition from the elastic case toward the RPP case. In addition it is in agreement with the major qualitative result of the finite element

calculations: the very large range of values of the indentation index X over which elasticity has a significant influence on the contact geometry.



a) Evolution of the shape ratio c versus the indentation index X .



b) Evolution of reduced hardness H/σ_0 versus the indentation index X .

Figure 4. Comparison of the results of the present model with the results of the **Johnson's cavity model** [4] and the results of the numerical simulation of conical indentation ($\theta = 70.3$ deg) with the finite element method under zero friction ($\bar{m} = 0$) and the maximal shear stress ($\bar{m} = 1$) [5].

§ 5. CONCLUSION

In this paper we improve the model of the expansion of the spherical cavity by building a velocity field whose components vary with the distance to the indenter apex r as $1/r^s$ with s a free parameter. By neglecting the material displacement and applying the upper bound theorems we calculate the more representative value of the free parameter s for elastic solid ($s = 1$) and RPP solid ($s = 4.35$). We analyse then the conical indentation of EPP solids characterised by the indentation index X and estimate the hardness H and the shape ratio c . The results of the model are in rather good agreement with the available exact results (elastic case) and the numerical ones (RPP and EPP cases). We obtain so a complete description from

elastic solid ($X < 3/2$) to RPP solid ($X > 1000$) of the evolution with the indentation index X of the velocity field, the extent of the plastic zone and the strain field. Further work is actually in progress in order to apply this approach to other problems related to indentation or scratch test.

APPENDIX: NOMENCLATURE

A	area of the projected contact surface (πr_c^2)
b	radius of the elastic-plastic boundary
B	ratio of the elastic-plastic boundary radius to the contact radius (b/r_c)
c^p	shape ratio (h_c/h)
E	Young's modulus
E^*	effective elastic modulus
$F(s)$	a function of the exponent s (relation (9))
h	penetration depth
h_c	contact depth
H	hardness (P/A)
H^*	reduced hardness (H/σ_0)
\bar{m}	Tresca's friction coefficient
p	exponent of the indenter profile: cone: $p = 1$; sphere: $p = 2$; flat punch: $p \infty$
P	indentation force
r_c	contact radius
ρ, ψ, z	polar coordinates
r, ϕ, ψ	spherical coordinates
R^{p-1}	quantity defining the indenter shape: $p = 1$: $\tan \theta$, $p = 2$: R = indenter radius
s	exponent of the velocity field
u	velocity
t	time
X	indentation index ($(E^*/\sigma_0) \tan \beta$)
V	indentation velocity (dh/dt)
W	work of deformation
\dot{W}	power of deformation
β	angle of inclination of the indenter at the edge of the contact
$\dot{\epsilon}$	strain rate tensor
$\bar{\dot{\epsilon}}$	generalised strain rate ($\sqrt{2\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}/3}$)
$\bar{\epsilon}$	generalised strain ($\int \bar{\dot{\epsilon}} dt$)
δ	number ($\pi(1-\nu)/2$)
2θ	apical angle of a conical indenter
μ	elastic shear modulus
σ_0	flow stress
τ	friction shear stress
ν	Poisson's ratio

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